Coloring problems

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Abstract

Who does not remember coloring as a child? It was a chance to let our imagination run free, guided by a single challenge: not to go outside the lines! Today, we have grown up, and those days feel like a distant memory. In this talk, we will attempt to revive this childhood pastime from a mathematical perspective. However, the goal is quite different. We want to determine the number of distinct colorings of an object made up of identical regions. At first glance, this may seem like a purely combinatorial problem. Now, if we color the object and then manipulate it without altering its structure, we still obtain the same coloring. Group actions then become an essential tool for counting these colorings. Concrete geometric examples will be explored to highlight the connection with abstract algebra and to develop an intuitive understanding of the underlying concepts.

Contents

1 Background

Imagine a two- or three-dimensional object made up of identical "cells" that can be colored using a given number of colors. At first glance, this type of problem seems to rely solely on combinatorial methods rather than on abstract algebra.

Consider a cube. We wish to color each of its faces using at most two colors. How many distinct colorings of the cube are possible? A first approach is to observe that each face can be assigned one of two possible colors, so the total number of colorings is $2^6 = 64$. However, this answer ignores the symmetries of the cube. Indeed, consider a first cube whose front face is green while all the other faces are blue, and a second cube whose right face is green while all the others are blue. By rotating the first cube, we obtain a cube identical to the second one.

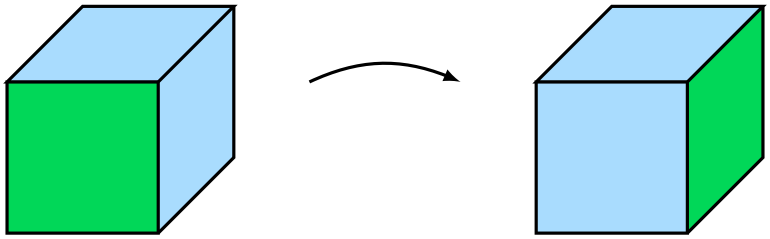

FIGURE 1: Two bicolored cubes identified through a rotation

Thus, the goal is not only to count the number of colorings of a given object, but also to identify those which can be obtained from a given coloring through the action of an isometry of the object onto itself. Therefore, group actions become an essential tool for solving this class of problems.

2 Colorings and group theory

2.1 Group action on the set of colorings

DEFINITION 2.1.1. Given a nonempty finite set E and a positive integer q, a **q-coloring** of E is a map from E to $\{1, ..., q\}$.

Notation 2.1.2. We denote by $C(E, q)$ the set of q-colorings of E.

Example 2.1.3. Consider a domino and assume that blue and green are the only available colors. We denote by A the first region of the domino and B the second.

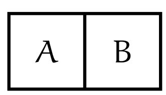

FIGURE 2: Naming the regions of the domino

Assume also that 1 represents the color blue and 2 the color green. In other words, $E = \{A, B\}$ and $q = 2$. Hence, we have

$$C(E,q) = \left\{ \begin{array}{ccc} E & \mapsto & \{1,2\} \\ A & \mapsto & 1 \\ B & \mapsto & 1 \end{array}, \begin{array}{ccc} E & \mapsto & \{1,2\} \\ A & \mapsto & 1 \\ B & \mapsto & 2 \end{array}, \begin{array}{ccc} E & \mapsto & \{1,2\} \\ A & \mapsto & 2 \\ B & \mapsto & 1 \end{array}, \begin{array}{ccc} E & \mapsto & \{1,2\} \\ A & \mapsto & 2 \\ B & \mapsto & 2 \end{array} \right\}.$$

Visually, it corresponds to the following colorings.

FIGURE 3: A first approach to determine all bicolored dominoes

The set E thus represents the set of regions of the object to be colored and q the number of available colors. We now wish to identify two colorings of the object whenever they differ by an isometry.

Notation 2.1.4. Let E be a nonempty finite set, q a positive integer and let G be a finite group acting on E. For every $g \in G$ and for every $f \in C(E, q)$, we denote by $g \cdot f$ the element of $C(E, q)$ defined for every $x \in E$ by

$$(g \cdot f)(x) = f(g^{-1} \cdot x).$$

Proposition 2.1.5. Let E be a nonempty finite set, q a positive integer and G a finite group acting on E. The map

$$\begin{array}{ccc} G \times C(E, q) & \to & C(E, q) \\ (g, f) & \mapsto & g \cdot f \end{array}$$

is a group action of G on $C(E, q)$.

Proof. We must check that the above map indeed defines a group action.

Let $f \in C(E, q)$. For every $x \in E$,

$$(1_G \cdot f)(x) = f(1_G^{-1} \cdot x) = f(x)$$

and therefore $1_G \cdot f = f$.

Now, let $g \in G$, $g' \in G$ and $f \in C(E, q)$. For every $x \in E$,

$$(g \cdot (g' \cdot f))(x) = (g' \cdot f)(g^{-1} \cdot x) = f(g'^{-1} \cdot (g^{-1} \cdot x)) = f((g'^{-1}g^{-1}) \cdot x) = f((gg')^{-1} \cdot x) = ((gg') \cdot f)(x).$$

Hence $(gg') \cdot f = g \cdot (g' \cdot f)$. This proves that G acts on $C(E, q)$, as claimed. ❏

2.2 Equivalence of two colorings

Definition 2.2.1. Let E be a nonempty finite set, q a positive integer and G a finite group acting on E.

We say that two colorings $f \in C(E, q)$ and $f' \in C(E, q)$ are **equivalent** if there exists $g \in G$ such that $f' = g \cdot f$ (that is, f and f' belong to the same orbit).

Notation 2.2.2. We denote by $C_G(E, q)$ the set of orbits of $C(E, q)$ under the action of G.

This corresponds well to the intuitive idea of identifying two colorings. If G is the group of isometries of the object to be colored, this definition simply states that two colorings are equivalent if one can be transformed into the other by moving the object in question.

Example 2.2.3. Let us go back to the domino introduced in Example 2.1.3. Consider the map $\tau : E \to E$ which sends A to B and B to A and denote by G the group $\{\mathrm{Id}_E, \tau\}$. The latter naturally acts on E. Then the colorings

$$\begin{array}{cccc} f_1 : & E & \to & \{1, 2\} \\ & A & \mapsto & 1 \\ & B & \mapsto & 1 \end{array} \quad \text{and} \quad \begin{array}{cccc} f_2 : & E & \to & \{1, 2\} \\ & A & \mapsto & 2 \\ & B & \mapsto & 2 \end{array}$$

are non-equivalent, but

$$\begin{array}{cccc} f_3 : & E & \to & \{1, 2\} \\ & A & \mapsto & 1 \\ & B & \mapsto & 2 \end{array} \quad \text{and} \quad \begin{array}{cccc} f_4 : & E & \to & \{1, 2\} \\ & A & \mapsto & 2 \\ & B & \mapsto & 1 \end{array}$$

are equivalent because $f_3 = \tau \cdot f_4$. Hence, we have

$$C_G(E, q) = \{[f_1], [f_2], [f_3]\}.$$

Visually, it corresponds to the following colorings.

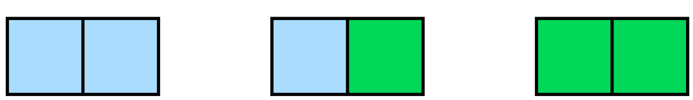

Figure 4: All bicolored dominoes

The goal is to determine the total number of pairwise non-equivalent colorings under the action of G, that is, the cardinality of $C_G(E, q)$. For this purpose, we apply Burnside's lemma, which gives a formula for the number of orbits of a group action.

Proposition 2.2.4 (Burnside's lemma). Let G be a finite group acting on a nonempty set X. Let Ω denote the set of orbits of the action.

$$|\Omega| = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|$$

where for all $g \in G$, $\mathrm{Fix}(g) = \{x \in X \mid g \cdot x = x\}$.

Proof. Consider the set $E = \{(g, x) \in G \times X \mid g \cdot x = x\}$. The goal is to calculate $|E|$ in two different ways. On the one hand

$$E = \bigcup_{x \in X} \{(g, x) \mid g \in G \text{ and } g \cdot x = x\} = \bigcup_{x \in X} \{g \in G \mid g \cdot x = x\} \times \{x\} = \bigcup_{x \in X} \mathrm{Stab}_G(x) \times \{x\}.$$

Since the previous union is disjoint, we have

$$|E| = \sum_{x \in X} |\mathrm{Stab}_G(x)||\{x\}| = \sum_{x \in X} |\mathrm{Stab}_G(x)|.$$

Now, for every $x \in X$, if $\mathcal{O}_x$ denotes the orbit of x under the action of G, then we know that $|\mathcal{O}_x| = \dfrac{|G|}{|\mathrm{Stab}_G(x)|}$. It follows that

$$|E| = \sum_{x \in X} \frac{|G|}{|\mathcal{O}_x|} = |G| \sum_{x \in X} \frac{1}{|\mathcal{O}_x|}.$$

Then the orbits of the action of G on X form a partition of X. Moreover, if $\omega \in \Omega$, then for every $x \in \omega$, we have $\mathcal{O}_x = \omega$. We deduce that

$$|E| = |G| \sum_{\omega \in \Omega} \sum_{x \in \omega} \frac{1}{|\mathcal{O}_x|} = |G| \sum_{\omega \in \Omega} \sum_{x \in \omega} \frac{1}{|\omega|} = |G| \sum_{\omega \in \Omega} |\omega| \frac{1}{|\omega|} = |G||\Omega|. \tag{$*$}$$

On the other hand

$$E = \bigcup_{g \in G} \{(g, x) \mid x \in X \text{ and } g \cdot x = x\} = \bigcup_{g \in G} \{g\} \times \{x \in X \mid g \cdot x = x\} = \bigcup_{g \in G} \{g\} \times \mathrm{Fix}(g).$$

Once again, the previous union being disjoint, we have

$$|E| = \sum_{g \in G} |\{g\}||\mathrm{Fix}(g)| = \sum_{g \in G} |\mathrm{Fix}(g)|. \tag{$**$}$$

Thus, equalities $(*)$ and $(**)$ yield $|G||\Omega| = \sum_{g \in G} |\mathrm{Fix}(g)|$. The formula then follows by dividing both sides by $|G|$. ❑

Corollary 2.2.5. With the notation of the previous proposition, we have

$$|\Omega| = \frac{1}{|G|} \sum_{x \in X} |\mathrm{Stab}_G(x)|.$$

Notation 2.2.6. If E is a nonempty finite set, q a positive integer, G a finite group acting on E, and g an element of G, we denote by $C(E, q)(g)$ the set of elements of $C(E, q)$ fixed by g:

$$C(E, q)(g) = \{f \in C(E, q) \mid g \cdot f = f\}.$$

Proposition 2.2.7. Let E be a nonempty finite set, q a positive integer, G a finite group acting on E and g an element of G. An element f of $C(E, q)$ belongs to $C(E, q)(g)$ if and only if it is constant on each g-orbit.

Proof. Suppose that $f \in C(E, q)(g)$, then $g \cdot f = f$. Equivalently, for all $x \in E$, we have $f(g^{-1} \cdot x) = f(x)$, so $f(g \cdot x) = f(x)$.

Fix $x \in E$. If ω denotes the g-orbit of x, then $\omega = \{g^m \cdot x \mid m \in \mathbb{Z}\}$. The preceding relations and a straightforward induction show that f is constant on ω.

Conversely, assume that f is constant on g-orbits. For every $x \in E$, the elements $g^{-1} \cdot x$ and x lie in the same g-orbit. As this holds for every $x \in E$, it follows that $g \cdot f = f$. Hence $f \in C(E, q)(g)$. ❑

Theorem 2.2.8. Let E be a nonempty finite set, q a positive integer and G a finite group acting on E. We have

$$|C_G(E,q)| = \frac{1}{|G|}\sum_{g\in G} q^{r(g)}$$

where for all $g \in G$, $r(g)$ is the number of g-orbits.

Proof. First, by Burnside's lemma,

$$|C_G(E,q)| = \frac{1}{|G|}\sum_{g\in G} |C(E,q)(g)|.$$

Let $g \in G$ and let $f \in C(E,q)$. By Proposition 2.2.7, $f \in C(E,q)(g)$ if and only if f is constant on each g-orbit. For each g-orbit, one may choose an arbitrary color and there are q possible choices. Moreover, these choices are independent. Therefore,

$$|C(E,q)(g)| = \underbrace{q \times \ldots \times q}_{\text{number of } g\text{-orbits}} = q^{r(g)}.$$

This completes the proof. ❏

3 Some classical coloring problems

3.1 The roulette wheel

Consider a roulette wheel divided into n sectors and q available colors. We identify two colorings whenever one can be transformed into the other by a rotation.

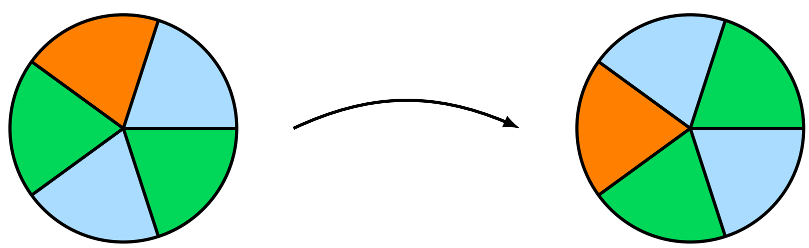

Figure 5: Two 5-sector, 3-color roulette wheels through a rotation by angle $\frac{2\pi}{5}$

The goal is to determine the number of distinct colorings. For this purpose, we identify each sector of the roulette wheel with a vertex of a regular n-gon.

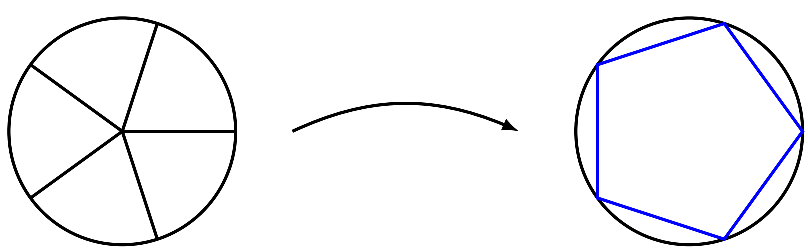

Figure 6: A 5-sector roulette wheel identified with a regular pentagon

Let $G = \langle r \rangle$ where r is the rotation about O (the center of the n-gon) through an angle equal to $\frac{2\pi}{n}$. Let E be the set of vertices of the n-gon. Then G acts naturally on E. Turning the roulette wheel corresponds to the action of G on E.

Now, the goal is to calculate $|C_G(E,q)|$. To start with, for every integer k satisfying $1 \leqslant k \leqslant n-1$, we have $r^k \neq \mathrm{Id}_E$ and $r^n = \mathrm{Id}_E$, hence $|G| = n$.

Let $g \in G$. We recall that a g-orbit is simply an orbit of E under the action of the cyclic subgroup $\langle g \rangle$. First, $|\langle g \rangle| = o(g)$. Second, the only planar rotation about O that fixes a point other than O is the identity on E. Equivalently, for every

$S \in E$, $\mathrm{Stab}_{\langle g\rangle}(S) = \{\mathrm{Id}_E\}$, and thus $|\mathrm{Stab}_{\langle g\rangle}(S)| = 1$. By Corollary 2.2.5,

$$r(g) = \frac{1}{|\langle g\rangle|}\sum_{S\in E}|\mathrm{Stab}_{\langle g\rangle}(S)| = \frac{1}{o(g)}\sum_{S\in E}1 = \frac{n}{o(g)}.$$

Next, Theorem 2.2.8 implies that

$$|C_G(E,q)| = \frac{1}{n}\sum_{g\in G} q^{\frac{n}{o(g)}}.$$

We may further refine the previous formula by grouping the elements according to their order. Indeed, by Lagrange's theorem, for every $g \in G$, $o(g)$ divides n. Moreover, for every positive divisor d of n, the number of elements of order d in G is equal to $\varphi(d)$, where φ is Euler's totient function.

Therefore, the number of distinct colorings of an n-sector roulette wheel with at most q colors is equal to

$$\boxed{|C_G(E,q)| = \frac{1}{n}\sum_{\substack{d|n\\ d>0}} \varphi(d) q^{\frac{n}{d}}.}$$

Remark 3.1.1. If $n = q = 3$, we have 11 distinct colorings of the wheel. We can even enumerate them.

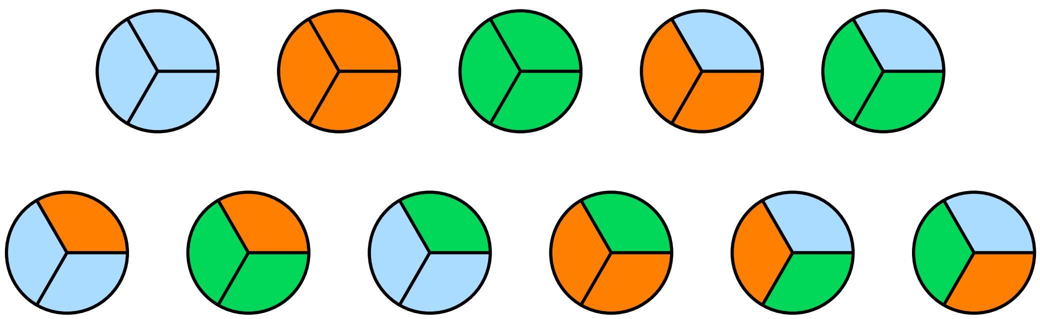

FIGURE 7: List of all 3-sector, 3-color roulette wheels

Of course, with the previous formula, we also obtain the same number of colorings:

$$|C_G(E,q)| = \frac{1}{3}\sum_{\substack{d|3\\ d>0}} \varphi(d)3^{\frac{3}{d}} = \frac{1}{3}(3^3+6) = 11.$$

When the values of n and q are small, it is easy to explicitly list all the possibilities. However, as n and q increase, the formula becomes essential.

3.2 The necklace

Consider a necklace with n beads of the same size. Suppose that q different colors are available. Two necklaces are identified if one can be obtained from the other by a rotation or a reflection.

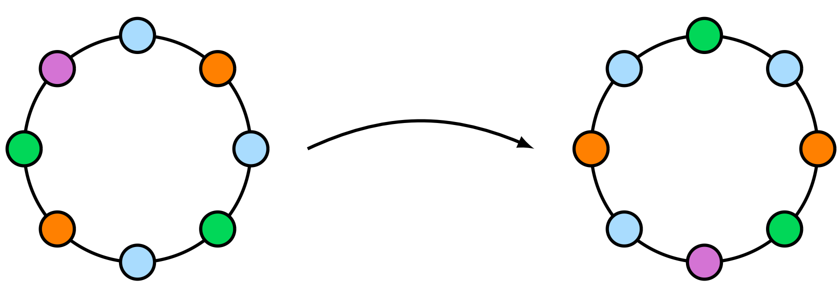

FIGURE 8: Two 8-bead, 4-color necklaces identified through a rotation by angle $\frac{3\pi}{4}$

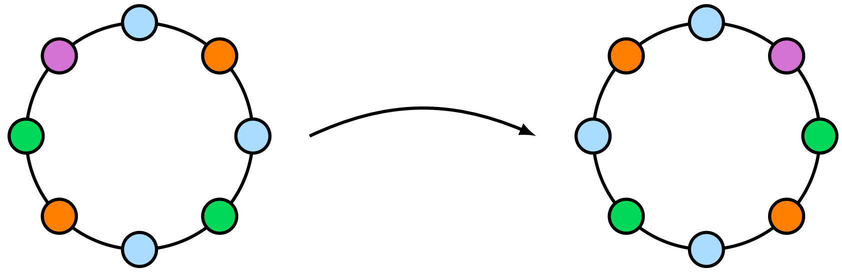

FIGURE 9: Two 8-bead, 4-color necklaces identified through a reflection across the vertical axis

Once again, we aim to determine the number of distinct colorings. Although this problem appears to be the same as the roulette wheel case, the presence of symmetries requires some modifications.

Identifying the beads of the necklace with the vertices of a regular n-gon, we find that the dihedral group D_n acts on the set of vertices of the n-gon. This group consists of the rotations about O (the center of the n-gon) through angles $\frac{2k\pi}{n}$, where $0 \leqslant k \leqslant n-1$, together with the n reflections whose axes pass through O. Therefore, $|D_n| = 2n$.

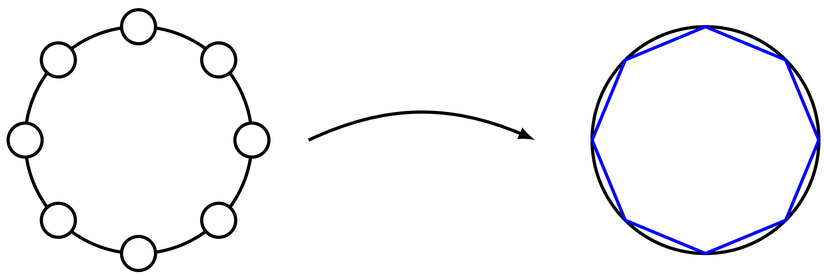

FIGURE 10: An 8-bead necklace identified with a regular octagon

From the results established in Section 3.1, we know that the number of orbits of a rotation of order d (where $d \mid n$) is $\frac{n}{d}$. The situation is different for reflections, as the number of orbits depends on the parity of n.

Case 1. Suppose that n is odd. Then $n = 2m - 1$ for some positive integer m.

Let s be a reflection. Its axis passes through exactly one vertex of the n-gon. Furthermore, s fixes this vertex and interchanges the remaining vertices in pairs.

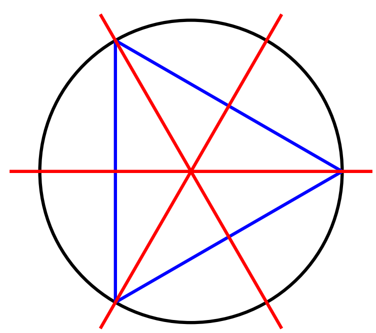

FIGURE 11: Reflections in the dihedral group D_3

Thus, each reflection has one orbit of size 1 and $m - 1 = \frac{n-1}{2}$ orbits of size 2. Hence, each of the n reflections has a total of $\frac{n+1}{2}$ orbits.

Case 2. Assume that n is even. Then $n = 2m$ for some positive integer m.

Among the reflections, m have an axis passing through two vertices, while the remaining m have an axis passing through no vertex. In the former case, the reflection fixes two vertices and interchanges the remaining $n-2$ vertices in pairs. In the latter case, it interchanges all n vertices in pairs and has no fixed points.

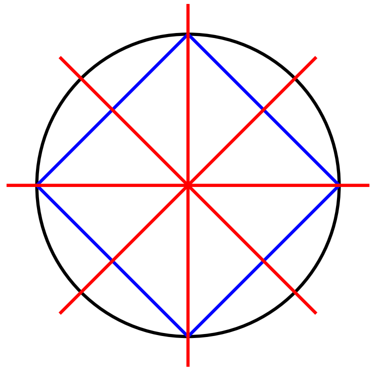

FIGURE 12: Reflections in the dihedral group D_4

Thus, we have

- $m = \frac{n}{2}$ reflections, each having two orbits of size 1 and $m-1$ orbits of size 2, that is, a total of $m+1 = \frac{n+2}{2}$ orbits.
- $m = \frac{n}{2}$ reflections, each having m orbits of size 2.

Therefore, the number of distinct colorings of an n-bead necklace with at most q colors is equal to

$$\boxed{|C_G(E,q)| = \begin{cases} \dfrac{1}{2n}\left(\displaystyle\sum_{\substack{d|n \\ d>0}} \varphi(d) q^{\frac{n}{d}} + n q^{\frac{n+1}{2}}\right) & \text{if } n \text{ is odd} \\ \dfrac{1}{2n}\left(\displaystyle\sum_{\substack{d|n \\ d>0}} \varphi(d) q^{\frac{n}{d}} + \dfrac{n}{2} q^{\frac{n+2}{2}} + \dfrac{n}{2} q^{\frac{n}{2}}\right) & \text{if } n \text{ is even} \end{cases}.}$$

Remark 3.2.1. If $n = 4$ and $q = 2$, we have 6 distinct colorings of the necklace. Here is the list.

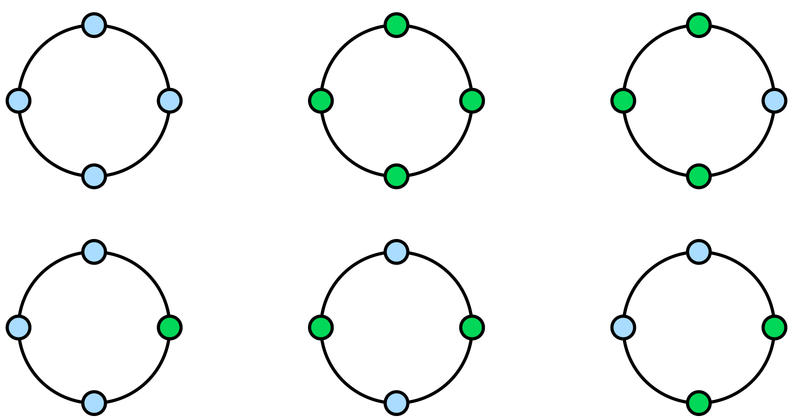

FIGURE 13: List of all 4-bead, 2-color necklaces

With the previous formula, we have

$$|C_G(E,q)| = \frac{1}{8}\left(\sum_{\substack{d|4 \\ d>0}} \varphi(d) 2^{\frac{4}{d}} + 2\times 2^3 + 2\times 2^2\right) = \frac{1}{8}(16+4+4+16+8) = 6.$$

As in Remark 3.1.1, it is tedious to explicitly list all possible colorings when n and q are large.

Reference

[Ber] Grégory Berhuy. *Algèbre : le grand combat*. Calvage & Mounet, 2020.